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AN ASYMPTOTIC ANALYSIS OF STATIONARY AND MOVING CRACKS WITH FRICTIONAL CONTACT ALONG BIMATERIAL INTERFACES AND IN HOMOGENEOUS SOLIDS

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Abstract—The plane strain and plane stress problem of a stationary or steadily moving crack with frictional sliding crack surface contact is investigated, with emphasis on the asymptotic structure of the crack tip fields. The crack is assumed to lie along the interface of an elastic anisotropic bimaterial with an aligned plane of symmetry, which covers special cases where the bimaterial is orthotropic or isotropic, or where the bimaterial becomes homogeneous. A full representation of the asymptotic fields around the interface crack is derived in terms of several arbitrary analytic functions, with explicit expressions for the singular crack tip stress and displacement fields given for a steadily propagating interface crack in an isotropic bimaterial, which are used to predict the direction of possible crack deviation from the interface. For a stationary crack, the singularity of the stresses can be, in general, stronger or weaker than $r^{-1/2}$ (where r is the distance to the crack tip) depending on the loading history, while for a steadily growing crack, the singularity must be weaker than $r^{-1/2}$, resulting in zero energy release rate at the crack tip. For bimaterials with orthotropic symmetries, the form of the singular stress field is found somewhat similar to that of the classic mode II problem. When these types of materials become homogeneous, and irrespective of the amount of friction between the contacting crack faces, the singular crack tip fields are identical to those of the classic mode II problem. Hence, the solutions are also governed by the conventional stress intensity factor K_{II}, implying a nonzero crack tip energy release rate, which is related to K_{II} in the usual manner. Implications of the above findings will be discussed.

1. INTRODUCTION

Cracks in homogeneous solids or along bimaterial interfaces play an important role in determining the behavior of materials and structures. Within the context of linear elastic fracture mechanics, the strength and toughness of a cracked solid can be characterized in terms of the crack tip energy release rate and its critical value, the fracture toughness of the material. In this connection, we note that, in evaluating the energy release rate and measuring the fracture toughness, it is commonly assumed, although implicitly, that the crack surfaces are not in contact. Although this assumption can be designed to hold under laboratory conditions, it will be violated in many real-life situations. Typical examples include (a) cases where cracks are subjected to combined compression and shear loadings, so that crack faces come into contact, and (b) cases where cracks lie along bimaterial interfaces under mostly shear loadings, so that sizeable contact zones emerge near the crack tip (Willis, 1972; Comninou and Schmueser, 1979; Gautesen and Dundurs, 1988). When the crack faces are rough and rugged, as they often are, friction will be generated when the contacting crack surfaces slide over each other.

The effect of friction between the sliding crack faces on the fracture behavior of a cracked structure can be argued in several ways. Physically, friction provides resistance to external loading and consumes energy, hence reduces crack tip stress intensity and the amount of energy available for fracture initiation and crack growth, making the structure appear tougher. Mathematically, friction couples the shear and normal traction components along the crack faces. This coupling may modify the structure of the crack tip stress and deformation fields, which may in turn alter the mathematical form of the conventional fracture parameter used for cases where frictional crack surface contact is assumed not to exist. This has special significance for interface cracks. As shown in this study for an interface crack between two dissimilar anisotropic solids under plane strain and plane stress conditions, the singularity of the crack tip fields will be weaker than $r^{-1/2}$ if the crack is growing under local steady-state conditions, where r is the distance to the crack tip. This means that, within the context of linear elastic fracture mechanics, the *theoretical* energy release rate will be zero at the crack tip, and any nonzero value obtained via a near-tip contour integral will be path-dependent. On one hand, this result implies that very little energy will be available for the growth of the interface crack when sliding friction is present, and it can explain why an energy release rate based fracture toughness value peaks when the loading is dominantly shear [for experimental observations of this phenomenon please refer to a review article by Hutchinson and Suo (1992)]. On the other hand, this result suggests that the energy release rate based fracture criterion in use today lacks a sound theoretical foundation for problems involving interface cracks with sliding friction, and it must be modified properly in order to be applicable to such interface crack problems.

As alluded to in the above, the work reported below investigates the asymptotic features of the crack tip fields for interface cracks with sliding frictional contact between the crack faces and provides a basis for further development in this and related areas. The findings of this work can have important applications in a number of areas. One implication of the findings can be seen from the brief discussion of the preceding paragraph regarding the basis of fracture criteria and the measure for the fracture toughness. For example, for the case of growing interface cracks with sliding friction in the contact zone, it can be claimed that nonzero energy release rate values obtained from finite element solutions near the crack tip are arbitrary and not meaningful, since converged finite element solutions will result in zero values. The crack tip fields obtained in this study also can play an important, and sometimes, essential role in computational and experimental studies of the subject. For example, because of the nonlinearity associated with possible large scale crack face contact under mixed mode loading conditions, numerical solutions of interfacial crack problems have been very difficult. One way to improve the convergence and accuracy of the numerical solutions is to incorporate the asymptotic features of the crack tip fields into the numerical codes. This approach has been very effective. It is interesting to note that even the success of analytical solutions often relies on the general understanding of the crack tip fields. Finally, it is noted that the availability and understanding of the asymptotic crack tip fields are critical to the proper interpretation of experimental measurements from cracked specimens, especially those from crack tip areas.

The asymptotic problem of interface cracks with sliding friction in the contact zone was first studied by Comninou (1977) and further analysed by Comninou and Dundurs (1980), for stationary interface cracks in dissimilar isotropic solids. Their asymptotic analyses, with the use of the common Williams' eigen-expansion technique (Williams, 1959), reveal that, depending on the loading history, the singularity of the crack tip stress field can be stronger or weaker than that of the classic mode II problem. Furthermore, if an interface crack is to deviate from the interface, it will be unlikely that it will veer into the solid with the higher elastic modulus (assuming equal Poisson's ratio). Based on the results of their studies, an additional conclusion can be made: when the two phases of the isotropic bimaterial have the same elastic constants, the asymptotic crack tip singular field will be identical to that of the classic mode II problem, with the same singularity and the same angular variations, regardless of the amount of friction.

In this paper, we will extend the previous analyses to deal with general two-dimensional problems (plane strain and plane stress) in anisotropic materials, and we will consider both stationary and steadily growing interface cracks, both with sliding friction in the contact zone. We will also examine the effect of friction on the asymptotic near-tip fields around cracks in homogeneous elastic solids. This work represents a further development on the subject of interface cracks in elastic solids, which progresses in part from some recent works by this author (Deng, 1993a,b,c). In particular, the reference Deng (1993a) extends the classic, interfacial crack tip fields (which assumes, in addition to isotropy, that cracks are always open and their faces are not in contact), to give the complete representation of the crack tip fields for the general case of coupled in-plane and out-of-plane deformations in arbitrary anisotropic bimaterials. Explicit Williams-type series expansions of the crack tip fields for stationary and steadily growing interface cracks in isotropic bimaterials are derived as special cases. The purpose of the reference Deng (1993c) is to deal with cases where the open crack assumption is not valid, such as when shear loadings are involved. Because of the complexity of the problem, it is assumed that the interface crack is in frictionless contact and is propagating steadily in an isotropic bimaterial. Explicit expressions for the crack tip fields and the energy release rate are given. The reference Deng (1993b) is concerned with the frictionless crack–surface contact situation in arbitrarily anisotropic solids, which is especially important for composite materials. General, coupled in-plane and out-of-plane deformations are considered and some unusual findings are obtained. The above results for interface cracks in frictionless contact can be used to model problems where constant friction exists between the crack surfaces, but not problems where Coulomb-type sliding friction is expected. Since in many practical situations Coulomb-type friction is in effect between the crack (interfacial or not) faces, it is important that we understand its effect on the crack tip fields, which motivated the current study.

The arrangement of the paper is as follows. In Section 2, the mathematical formulation of the problem is outlined. In Section 3, the complete solutions of the asymptotic crack tip fields are derived for a general anisotropic bimaterial. In Section 4, explicit specifications for steady-state crack growth in an isotropic bimaterial are given. In Section 5, particular results for cracks in homogeneous anisotropic solids are observed, with focus on solids with orthotropic symmetries. In Section 6, the energy release rate is defined and evaluated in terms of a contour integral, and its general path dependence is discussed. Findings of this section will be used to interpret the results of a study by Stringfellow and Freund (1993) regarding the effect of interfacial friction on the delamination of a compressed thin film from its substrate. In Section 7, the main findings of this analytic study are summarized.

For the convenience of the reader, the symbols in this paper are kept similar to those in Deng (1993a,b,c) to achieve certain uniformity in formulation. To avoid unnecessary repetition, the number of governing equations is kept to a minimum and many intermediate steps of the solution are omitted. For more details, the reader is referred to the author's previous papers and other related publications in the literature as given in the references.

2. MATHEMATICAL FORMULATION

As illustrated in Fig. 1, we consider a semi-infinite crack between the interface of two dissimilar, elastic, anisotropic half-planes, where the coordinate system is assumed to translate with the crack in the case of steady-state crack growth. To separate the in-plane deformation from the out-of-plane deformation, hence considering only plane strain or plane stress problems in the chosen coordinate system, we require that the anisotropic materials occupying the half-planes are so aligned that the xy-plane becomes a mirror plane.

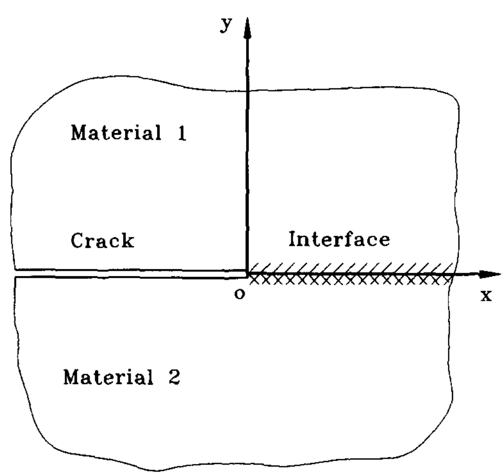

Fig. 1. An interface crack between two dissimilar elastic anisotropic half-planes.

Under the above settings, the stress and displacement fields for each of the half-planes can be fully represented by a complex function formulation due to Eshelby *et al.* (1953) and Stroh (1958, 1962), which can be written, for in-plane deformations only, as

$$\begin{aligned}\mathbf{u} &= 2\,\mathrm{Re}\,[\mathbf{Af}(z)]\\ \mathbf{t} &= 2\,\mathrm{Re}\,[\mathbf{Lf}'(z)]\\ \mathbf{s} &= -2\,\mathrm{Re}\,[\mathbf{L\Gamma f}'(z)-\rho v^2\mathbf{Af}'(z)].\end{aligned} \tag{1}$$

In the above, $\mathbf{u} = (u_x, u_y)$ is the displacement vector; $\mathbf{t} = (\sigma_{xy}, \sigma_{yy})$ and $\mathbf{s} = (\sigma_{xx}, \sigma_{xy})$ are stress vectors; the symbol Re denotes the real part of what follows; ρ is the mass density of the material; v is the speed of crack growth ($v = 0$ for stationary cracks) and is assumed here to be smaller than c_R, the bimaterial's smaller Rayleigh wave speed; and $\mathbf{f}(z) = [f_1(z_1), f_1(z_2)]$ is a vector composed of analytic functions, with $\mathbf{f}'(z) = [\mathrm{d}f_1/\mathrm{d}z_1, \mathrm{d}f_2/\mathrm{d}z_2]$, and the complex variables z_1 and z_2 and matrices $\mathbf{A}$, $\mathbf{L}$ and $\mathbf{\Gamma}$ are related to an eigenvalue problem described below. Assuming plane strain conditions, the eigenvalue problem is stated as $\mathbf{Pa} = \mathbf{0}$, where $\mathbf{a}$ is a vector and $\mathbf{P}$ is a symmetric matrix with components related to the eigenvalue p and the material's fourth order elasticity tensor C_{ijkl} as

$$\begin{aligned}P_{11} &= (C_{1111}-\rho v^2)+2C_{1112}p+C_{1212}p^2\\ P_{12} &= C_{1112}+(C_{1122}+C_{1212})p+C_{1222}p^2\\ P_{22} &= (C_{1212}-\rho v^2)+2C_{1222}p+C_{2222}p^2.\end{aligned} \tag{2}$$

When v is less than c_R, the eigenvalues or the roots of the equation $P_{11}P_{22}-P_{12}^2 = 0$ for the unknown p cannot be real (Eshelby *et al.*, 1953). Without loss of generality, the eigenvalues can be denoted by, say, p_1 and p_2 and their complex conjugates, where p_1 and p_2 have positive imaginary parts. The two eigenvectors associated with p_1 and p_2 respectively, can now be used as the two columns of the matrix $\mathbf{A}$, and matrix $\mathbf{L}$ can be generated from $\mathbf{A}$, p_1 and p_2 through

$$\mathbf{L}_{ij} = \sum_{k=1}^{k=2} [C_{i2k1}+p_j(C_{i2k2}-\rho v^2)]A_{kj} \quad (i,j = 1,2). \tag{3}$$

The matrix $\mathbf{\Gamma}$ is diagonal, with p_1 and p_2 as its two diagonal terms. The complex variables z_1 and z_2 are defined as $z_1 = x+p_1y$ and $z_2 = x+p_2y$, respectively. In the above formulation, the eigenvalues p_i ($i = 1,2$) are assumed to be distinct, which makes the matrices $\mathbf{A}$ and $\mathbf{L}$ nonsingular. Solutions for degenerate cases where p_i are not distinct can be obtained by taking appropriate limits of the final solutions for $\mathbf{u}$, $\mathbf{t}$ and $\mathbf{s}$, or derived from relevant complex function formulations [see e.g. Suo (1990)]. In the case of plane stress, the components of the elasticity tensor C_{ijkl} used in the above operations must be replaced by $(C_{ijkl}-C_{33ij}C_{33kl}/C_{3333})$.

For the purpose of easy algebraic manipulations, we can rewrite the expressions for $\mathbf{u}$ and $\mathbf{t}$ from eqn (1) as:

$$\begin{aligned}\mathbf{u} &= 2\,\mathrm{Im}\,[\mathbf{Bh}(z)]\\ \mathbf{t} &= 2\,\mathrm{Re}\,[\mathbf{h}'(z)]\end{aligned} \tag{4}$$

with

$$\mathbf{h}(z) = \mathbf{Lf}(z), \quad \mathbf{B} = \mathrm{i}\mathbf{AL}^{-1} \tag{5}$$

where $\mathrm{i} = \sqrt{-1}$ and symbol Im denotes the imaginary parts of what follows.

To eliminate ambiguity in the following, explicit subscripts 1 and 2 will be reserved, and used whenever deemed necessary, to signify quantities that are associated with, respectively, the upper and lower half-planes. Generic notations, unless otherwise specified, will still be used to denote quantities that are applicable to both materials.

In completing the mathematical formulation of the problem, we list here the boundary conditions to be satisfied by the analytic functions. Along the positive x-axis, the tractions and displacements are fully continuous across the interface:

$$\mathbf{t}_1 = \mathbf{t}_2, \quad \mathbf{u}_1 = \mathbf{u}_2 \tag{6}$$

and along the negative x-axis, the slip conditions are enforced between the sliding crack faces:

$$(\sigma_{xy})_1 = (\sigma_{xy})_2 = -\mu_k(\sigma_{yy})_1, \quad (\sigma_{yy})_1 = (\sigma_{yy})_2 \leqslant 0, \quad (u_y)_1 = (u_y)_2 \tag{7}$$

where μ_k equals the positive or negative value of the coefficient of kinetic friction depending on the relative sliding direction of the two crack faces. Specifically, $\mu_k > 0$ when $\mathrm{d}(u_x)_1/\mathrm{d}t - \mathrm{d}(u_x)_2/\mathrm{d}t > 0$ and $\mu_k < 0$ when $\mathrm{d}(u_x)_1/\mathrm{d}t - \mathrm{d}(u_x)_2/\mathrm{d}t < 0$, where $\mathrm{d}(\ )/\mathrm{d}t$ denotes time derivative.

3. GENERAL SOLUTION

In this section we derive the general form of $\mathbf{h}'(z)$ that satisfies the conditions set forth in eqns (6) and (7), which can be integrated with respect to z to yield $\mathbf{h}(z)$. Once $\mathbf{h}'(z)$ is obtained, it must be converted to vector $\mathbf{f}(z)$ according to the relation in (5), with z replaced by z_i ($i = 1, 2$) for its ith component. Substitution of $\mathbf{f}(z)$ into (1) then generates a complete representation of the crack tip stress and deformation fields.

To facilitate the derivation, we note that the equations in (6) and (7) can be recast into:

$$\begin{gathered} \mathbf{t}_1 = \mathbf{t}_2 \quad (y = 0, -\infty < x < \infty) \\ \mathbf{u}_1 = \mathbf{u}_2 \quad (y = 0, 0 < x < \infty) \\ (\sigma_{xy})_1 = -\mu_k(\sigma_{yy})_1, \quad (u_y)_1 = (u_y)_2 \quad (y = 0, -\infty < x < 0). \end{gathered} \tag{8}$$

Using the standard concept of analytic continuation, with details omitted here, we can show from the first line of (8) that the following is true over the whole plane:

$$\mathbf{h}_1'(z) - \bar{\mathbf{h}}_2'(z) = \mathbf{h}_2'(z) - \bar{\mathbf{h}}_1'(z) = \mathbf{g}(z) \tag{9}$$

where the vector $\mathbf{g}(z)$ is composed of entire functions and is restricted by $\mathbf{g}(z) = -\bar{\mathbf{g}}(z)$ but is otherwise arbitrary. [Here we understand that $\bar{\mathbf{g}}(z)$ denotes the complex conjugate of $\mathbf{g}(\bar{z})$, as implied by the overbar.] We further conclude, from the second line of (8), that the following holds everywhere on the whole plane except along the negative x-axis:

$$\mathbf{B}_1\mathbf{h}_1'(z) + \bar{\mathbf{B}}_2\bar{\mathbf{h}}_2'(z) = \mathbf{B}_2\mathbf{h}_2'(z) + \bar{\mathbf{B}}_1\bar{\mathbf{h}}_1'(z). \tag{10}$$

We note that eqns (9) and (10) are sufficient for expressing $\bar{\mathbf{h}}_1'(z)$, $\mathbf{h}_2'(z)$ and $\bar{\mathbf{h}}_2'(z)$ in terms of $\mathbf{h}_1'(z)$ and $\mathbf{g}(z)$, implying that we only need to concentrate on the solution for $\mathbf{h}_1'(z)$, which is what we hope to achieve from the last line of (8). Before we proceed, we find it convenient to introduce two hermitian matrices, $\mathbf{H}$ and $\mathbf{G}$, which are defined by

$$\mathbf{H} = \mathbf{B}_1 + \bar{\mathbf{B}}_2, \quad \mathbf{G} = \bar{\mathbf{B}}_1 - \bar{\mathbf{B}}_2. \tag{11}$$

It can be shown that when the crack speed is less than the smaller of the two Rayleigh wave speeds of the bimaterial, $\mathbf{H}$ is positive definite. Now from the last line of (8) and by making use of the relations derived from eqns (9) and (10), we obtain, for $y = 0$, $-\infty < x < 0$, the following nonhomogeneous Hilbert problem:

$$\mathbf{U}\mathbf{h}_1'^{+} + \mathbf{V}\mathbf{h}_1'^{-} = \mathbf{W}\mathbf{g}(x) \tag{12}$$

where superscripts $+$ and $-$ mean that the argument of $\mathbf{h}_1'(z)$ is evaluated just above and below the x-axis, respectively, and the components of the matrices $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are given by:

$$\begin{gathered}
U_{11} = |\mathbf{H}|, \quad U_{12} = \mu_k|\mathbf{H}|, \quad U_{21} = H_{21}, \quad U_{22} = H_{22}, \\
V_{11} = H_{11}H_{22} - H_{21}^2 + \mu_k H_{11}(H_{21} - H_{12}), \\
V_{12} = H_{22}(H_{12} - H_{21}) + \mu_k(H_{11}H_{22} - H_{12}^2), \\
V_{21} = -H_{21}, \quad V_{22} = -H_{22}, \\
W_{11} = |\mathbf{H}| - G_{11}H_{22} + G_{21}H_{21} + \mu_k(G_{11}H_{12} - G_{21}H_{11}), \\
W_{12} = G_{22}H_{21} - G_{12}H_{22} + \mu_k(|\mathbf{H}| - G_{22}H_{11} + G_{12}H_{12}), \\
W_{21} = W_{22} = 0,
\end{gathered} \tag{13}$$

where $|\mathbf{H}|$ stands for the determinant of $\mathbf{H}$. The general form of $\mathbf{h}_1'(z)$ is given by the general solution for eqn (12), which consists of a particular part and a homogeneous part. It is easy to see that eqn (12) has an infinite number of particular solutions, and it can be shown that a simple one coincides with the solution of this equation:

$$(\mathbf{H} + \bar{\mathbf{H}})\mathbf{h}_1'(z) = (\mathbf{H} - \bar{\mathbf{G}})\mathbf{g}(z) \tag{14}$$

which can be put in the form of (Deng, 1993b)

$$\mathbf{h}_1'(z) = [\mathbf{I} + (\operatorname{Re}\mathbf{B}_2)^{-1}\operatorname{Re}\mathbf{B}_1]^{-1}\mathbf{g}(z) \tag{15}$$

where $\mathbf{I}$ is the identity matrix. The corresponding particular solution for $\mathbf{h}_2'(z)$ can be obtained from (15) by switching $\mathbf{B}_1$ and $\mathbf{B}_2$. It is interesting to note that this particular solution is identical in form to that for an interface crack without contact (Deng, 1993a), which is because it represents a crack tip field that does not violate any boundary conditions along the x-axis for both types of interface crack problems. This can be seen from the fact that this particular solution produces displacements that are fully continuous across the whole x-axis and stresses that give rise to zero tractions along the bonded line as well as the crack faces.

To arrive at the homogeneous part of the general solution for $\mathbf{h}_1'(z)$, we note that we can express the unknown vector in terms of the two eigenvectors of the following eigenvalue problem

$$\mathbf{U}\mathbf{q} = \lambda\mathbf{V}\mathbf{q} \tag{16}$$

which has eigenvalues $\lambda = e^{-i2\pi\delta}$ and $\lambda^* = -1$ and the corresponding eigenvectors

$$\begin{gathered}
\mathbf{q} = \left(1, \quad -\frac{H_{21}}{H_{22}}\right) \\
\mathbf{q}^* = \left(\frac{i\operatorname{Im}H_{21}}{H_{11}} - \mu_k\left[1 - \frac{H_{12}\operatorname{Re}H_{21}}{H_{11}H_{22}}\right], \quad \frac{i\mu_k\operatorname{Im}H_{21}}{H_{22}} + \left[1 - \frac{H_{21}\operatorname{Re}H_{21}}{H_{11}H_{22}}\right]\right)
\end{gathered} \tag{17}$$

where δ is a real number, which can be determined from

$$\tan \pi\delta = \mu_k \beta, \quad (-1/2 < \delta < 1/2)$$
$$\beta = \mathrm{Im}\,(H_{21})/[H_{22} - \mu_k \,\mathrm{Re}\,(H_{21})]. \tag{18}$$

With such an arrangement, the homogeneous part of the general solution can be written as:

$$\mathbf{h}_1'(z) = \frac{z^\delta \xi(z)}{2\sqrt{\pi z}}\mathbf{q} + \xi^*(z)\mathbf{q}^* \tag{19}$$

where $\xi(z)$ and $\xi^*(z)$ are two arbitrary entire functions which are real-valued when the argument z is replaced by a real number. It can be shown, noting that $\mathbf{Hq} = \bar{\mathbf{H}}\bar{\mathbf{q}}$ and $\mathbf{Hq}^* = \bar{\mathbf{H}}\bar{\mathbf{q}}^*$, that the corresponding homogeneous solution for $\mathbf{h}_2'(z)$ of the lower half-plane can be generated from that for $\mathbf{h}_1'(z)$ by replacing $\mathbf{q}$ and $\mathbf{q}^*$ in (19) with their complex conjugates.

We recall that the general solution is obtained by combining the particular solution with the homogeneous one, hence it involves four entire functions, namely $\xi(z)$ and $\xi^*(z)$ plus the two component functions in $\mathbf{g}(z)$. These functions can be expanded into Taylor series at the crack tip $z = 0$, where the coefficients for $\xi(z)$ and $\xi^*(z)$ are real-valued and those for $\mathbf{g}(z)$ are purely imaginary. Substitution of these series in (1) will then generate a complete Williams-type series expansion for the crack tip stress and deformation fields. It is worth pointing out that the series from $\mathbf{g}(z)$ will generate *two* sets of terms with integer powers of r, the distance to the crack tip. These terms, as discussed previously, yield continuous displacements and *zero* tractions across the whole x-axis, and are present in crack tip field expansions for cracks with various crack surface conditions (e.g. without contact, with frictionless contact or with frictional contact). Similarly, it is noted that the series from $\xi^*(z)$ will generate *one* set of terms with integer powers of r, which give rise to continuous displacements and tractions (not necessarily zero) across the whole x-axis. In contrast, the series from $\xi(z)$ will generate *one* set of terms with non-integer powers of r, with the first term producing the only singular part of the crack tip fields, which can be represented by

$$\mathbf{h}_1'(z) = z^\delta K\mathbf{q}/2\sqrt{2\pi z}$$
$$\mathbf{h}_2'(z) = z^\delta K\bar{\mathbf{q}}/2\sqrt{2\pi z} \tag{20}$$

where K is the stress intensity factor. From (20), the traction vector along the bonded line is found to be

$$(\sigma_{xy}, \sigma_{yy}) = \frac{r^{-1/2+\delta}K}{\sqrt{2\pi}}\left(1, \quad -\frac{\mathrm{Re}\,H_{21}}{H_{22}}\right) \tag{21}$$

from which it is easy to see that the stress intensity factor K can be defined with σ_{xy} through $K = \lim_{r\to 0}[(2\pi)^{1/2}\sigma_{xy}r^{1/2-\delta}]$ evaluated along the bonded line. For an orthotropic bimaterial, the normal traction will be zero since $\mathrm{Re}\,H_{21} = 0$. In this case, the traction along the interface has the same form as that for a crack in a homogeneous material under mode II conditions, except that the singularity here is different. As such, it is reasonable to speculate that crack growth along the interface must be driven by shear stress there.

It is seen that the strength of the singularity represented by (20) depends on the sign of the singularity index δ, namely it will be weaker than $r^{-1/2}$ if $\delta > 0$ and stronger than $r^{-1/2}$ if $\delta < 0$. To gain better insight about the sign of δ, let us examine the normal traction σ_{yy} and sliding displacement $\Delta u_x \equiv (u_x)_1 - (u_x)_2$ along the crack surfaces, which are found to be

$$\sigma_{yy} = -r^{-1/2+\delta}(\beta K)\cos(\pi\delta)/\sqrt{2\pi}$$

$$\Delta u_x = 2r^{1/2+\delta}K\cos(\pi\delta)|\mathbf{H}|/(1+2\delta)H_{22}\sqrt{2\pi}. \tag{22}$$

Because of the positive definiteness of the matrix $\mathbf{H}$ and the particular interval $-1/2 < \delta < 1/2$, quantities $|\mathbf{H}|$, H_{22} and $\cos(\pi\delta)$ in the above expressions will be positive. Then from the crack surface contact requirement $\sigma_{yy} \leqslant 0$, we must have $K\beta \geqslant 0$. Since the sign of the friction coefficient μ_k is, by definition, the same as that of the time derivative of Δu_x or $\mathrm{d}\Delta u_x/\mathrm{d}t$, it is clear from eqn (18) that the sign of δ is the same as that of the product of $\mathrm{d}\Delta u_x/\mathrm{d}t$ and β, thus depending on the deformation or loading history as well as the mismatch parameter β of the bimaterial system.

For a stationary interface crack, it is easy to see from (22) that the sign of $\mathrm{d}\Delta u_x/\mathrm{d}t$ is equal to the sign of $\mathrm{d}K/\mathrm{d}t$. If the magnitude of K increases monotonically from zero, the sign of $\mathrm{d}K/\mathrm{d}t$ will be the same as that of K. In this case, the product $K\beta$ is always positive, as required by the crack surface contact requirement discussed earlier, hence δ will always be positive. It can be argued then that the crack tip field singularity will be weaker than $r^{-1/2}$ if an interface crack with frictional contact is loaded monotonically from the start.

For an interface crack that is growing steadily or whose crack tip field is in a local steady state, the steady-state approximation $\mathrm{d}(\ )/\mathrm{d}t = -v\partial(\ )/\partial x$ can be used to derive, from eqn (22) and noting that $r = -x$ along the crack flank, the following relation

$$\frac{\mathrm{d}\Delta u_x}{\mathrm{d}t} = \frac{vr^{-1/2+\delta}K\cos(\pi\delta)|\mathbf{H}|}{H_{22}\sqrt{2\pi}} \tag{23}$$

which holds near the crack tip. It is clear from (23) that the sign of $\mathrm{d}\Delta u_x/\mathrm{d}t$ is the same as that of K, hence the sign of δ will be the same as that of the product $K\beta$, which again is positive as required by the crack surface contact condition. Hence we conclude that the crack tip field singularity will be weaker than $r^{-1/2}$ for a moving interface crack with frictional crack surface contact, if the near-tip region is in a local steady state. A significant implication of this weaker singularity is that the conventional strain energy release rate G will be zero at the crack tip. This observation will be discussed further in Section 6 in connection with a recent study on the delamination of compressed thin films by Stringfellow and Freund (1993).

4. CRACK GROWTH IN ISOTROPIC BIMATERIALS

The solution presented in the preceding section applies to a stationary or growing crack along the interface of a general anisotropic bimaterial with the xy-plane as the aligned plane of symmetry. The solution involves, for each material, several matrices, such as $\mathbf{A}$ and $\mathbf{L}$, that pertain to an eigenvalue problem for the material, and are not given explicitly. To overcome certain inconvenience created by the implicit formulae for a general anisotropic bimaterial in discussing the results of the stress and displacement fields, we will specify the general results here for the case of an interface crack growing steadily in an isotropic bimaterial.

The complex function formulation presented in Section 2 for a generic anisotropic material can be reduced to a particular explicit form given by Radok (1956) for steady plane motion problems in isotropic solids. This can be done by redefining, respectively, the analytic function vector $\mathbf{f}(z)$ with $(-\phi/2G, \mathrm{i}\psi/2G)$, the complex variables z_1 and z_2 with z_l and z_s, and the eigenvalues p_1 and p_2 with α_l and α_s. The resulting formulae for stresses and displacements are

$$\sigma_{xx} = -\mathrm{Re}\,[(1+2\alpha_l^2-\alpha_s^2)\phi'(z_l)+2\alpha_s\psi'(z_s)]$$

$$\sigma_{yy} = \mathrm{Re}\,[(1+\alpha_s^2)\phi'(z_l)+2\alpha_s\psi'(z_s)]$$

$$\sigma_{xy} = \mathrm{Im}\,[2\alpha_l\phi'(z_l)+(1+\alpha_s^2)\psi'(z_s)] \tag{24}$$

$$u_x = -\mathrm{Re}\,[\phi(z_l)+\alpha_s\psi(z_s)]/G$$

$$u_y = \mathrm{Im}\,[\alpha_l\phi(z_l)+\psi(z_s)]/G. \tag{25}$$

In the above, G denotes the material's elastic shear modulus (rather than the energy release rate used earlier and later), and the constants α_l and α_s are related to the crack speed v and the material's longitudinal and shear wave speeds, c_l and c_s, respectively, through

$$\alpha_l = \sqrt{1-(v/c_l)^2}, \quad \alpha_s = \sqrt{1-(v/c_s)^2}$$

$$c_s = \sqrt{G/\rho}, \quad c_l = \sqrt{(k+1)/(k-1)}c_s \tag{26}$$

where $k = (3-v)/(1+v)$ in plane stress and $(3-4v)$ in plane strain. The matrix $\mathbf{H}$ for a crack growing along the interface of an isotropic bimaterial system is given by

$$\mathbf{H} = \begin{bmatrix} H_{11} & -\mathrm{i}H_{12}^* \\ \mathrm{i}H_{12}^* & H_{22} \end{bmatrix}. \tag{27}$$

With the definition of $D = 4\alpha_l\alpha_s-(1+\alpha_s^2)^2$ the components in (27) can be expressed as

$$H_{11} = [\alpha_s(1-\alpha_s^2)/GD]_1+[\alpha_s(1-\alpha_s^2)/GD]_2,$$

$$H_{22} = [\alpha_l(1-\alpha_s^2)/GD]_1+[\alpha_l(1-\alpha_s^2)/GD]_2,$$

$$H_{12}^* = [(2\alpha_l\alpha_s-1-\alpha_s^2)/GD]_1-[(2\alpha_l\alpha_s-\alpha_s^2)/GD]_2. \tag{28}$$

The eigenvectors in (17) are now simplified to

$$\mathbf{q} = (1, -\mathrm{i}\Gamma), \quad \mathbf{q}^* = (\mathrm{i}\Pi-\mu_k, 1+\mathrm{i}\mu_k\Gamma) \tag{29}$$

where $\Gamma = H_{12}^*/H_{22}$ and $\Pi = H_{12}^*/H_{11}$, both of which become zero when the bimaterial becomes homogeneous. The general solution for $\phi(z_l)$ and $\psi(z_s)$ for the complete interfacial crack tip fields can be generated from that for $\mathbf{h}(z)$ in (15) and (19). Noting

$$\begin{Bmatrix} \phi' \\ \psi' \end{Bmatrix} = \begin{bmatrix} 4\mathrm{i}\alpha_s/D & -2(1+\alpha_s^2)/D \\ -2\mathrm{i}(1+\alpha_s^2)/D & 4\alpha_l/D \end{bmatrix}\mathbf{h}' \tag{30}$$

and with certain rearrangements and redefinitions, we list here the expressions for $\phi'(z_l)$ and $\psi'(z_s)$ for the upper half-plane:

$$\phi'(z_l) = \frac{\mathrm{i}[2\alpha_s+\Gamma(1+\alpha_s^2)]}{D\sqrt{2\pi z_l}}z_l^\delta\xi(z_l) - \frac{2[2\alpha_s\Pi+(1+\alpha_s^2)]+2i\mu_k[2\alpha_s+\Gamma(1+\alpha_s^2)]}{D}\xi^*(z_l)$$
$$+\frac{[2\alpha_s\eta(1+\omega_l)-(1+\alpha_s^2)(1+\omega_s)]\zeta(z_l)}{D(1+\omega_l)(1+\omega_s)}+\frac{[2\alpha_s\eta(1+\omega_l)+(1+\alpha_s^2)(1+\omega_s)]\bar{\zeta}(z_l)}{D(1+\omega_l)(1+\omega_s)} \tag{31}$$

$$\psi'(z_s) = -\frac{\mathrm{i}[2\alpha_l\Gamma+(1+\alpha_s^2)]}{D\sqrt{2\pi z_s}}z_s^\delta\xi(z_s) + \frac{2[2\alpha_l+\Pi(1+\alpha_s^2)]+2i\mu_k[2\alpha_l\Gamma+(1+\alpha_s^2)]}{D}\xi^*(z_s)$$
$$+\frac{[2\alpha_l(1+\omega_s)-\eta(1+\alpha_s^2)(1+\omega_l)]\zeta(z_s)}{D(1+\omega_l)(1+\omega_s)}-\frac{[2\alpha_l(1+\omega_s)+\eta(1+\alpha_s^2)(1+\omega_l)]\bar{\zeta}(z_s)}{D(1+\omega_l)(1+\omega_s)} \tag{32}$$

which can be integrated, with respect to z_l and z_s respectively, to yield $\phi(z_l)$ and $\psi(z_s)$. In the above, the exponent δ now can be determined from $\tan(\pi\delta) = \mu_k\Gamma$, and the mismatch parameters η, ω_l and ω_s are given by

$$\eta = \sqrt{H_{22}/H_{11}}, \quad \omega_l = \frac{[\alpha_l(1-\alpha_s^2)/GD]_1}{[\alpha_l(1-\alpha_s^2)/GD]_2}, \quad \omega_s = \frac{[\alpha_s(1-\alpha_s^2)/GD]_1}{[\alpha_s(1-\alpha_s^2)/GD]_2}. \tag{33}$$

We point out that the new arbitrary entire function $\zeta(z)$ in (31) and (32) is derived from $\mathbf{g}(z)$. To obtain solution of the crack tip fields for the lower half-plane, one only need replace Γ with $-\Gamma$, Π with $-\Pi$, ω_l with ω_l^{-1}, and ω_s with ω_s^{-1} in eqns (31) and (32).

As discussed earlier in Section 3, a complete Williams-type series expansion of the interface crack tip fields can be generated by expanding the entire functions in (31) and (32) into Taylor series at the origin. We note that the coefficients of the series from $\xi(z)$ and $\xi^*(z)$ will be real, while those from $\zeta(z)$ will be complex. The singular crack tip fields derive from the first term of the series expansion for $\xi(z)$, which correspond to, for the upper half-plane,

$$\phi'(z_l) = \frac{\mathrm{i}[2\alpha_s+\Gamma(1+\alpha_s^2)]z_l^\delta K}{D\sqrt{2\pi z_l}}, \quad \psi'(z_s) = -\frac{\mathrm{i}[2\alpha_l\Gamma+(1+\alpha_s^2)]z_s^\delta K}{D\sqrt{2\pi z_s}} \tag{34}$$

where K is the stress intensity factor. Substitution of (34) into (24) and (25) then yields the following explicit expressions for the singular crack tip stress and displacement field variations of the upper half-plane ($0 \leqslant \theta_l, \theta_s \leqslant 180^\circ$):

$$\sigma_{xx} = \frac{K}{D\sqrt{2\pi}}\{-(1+2\alpha_l^2-\alpha_s^2)[2\alpha_s+\Gamma(1+\alpha_s^2)]r_l^{-1/2+\delta}\sin(1/2-\delta)\theta_l$$
$$+2\alpha_s[2\alpha_l\Gamma+(1+\alpha_s^2)]r_s^{-1/2+\delta}\sin(1/2-\delta)\theta_s\},$$

$$\sigma_{yy} = \frac{K}{D\sqrt{2\pi}}\{(1+\alpha_s^2)[2\alpha_s+\Gamma(1+\alpha_s^2)]r_l^{-1/2+\delta}\sin(1/2-\delta)\theta_l$$
$$-2\alpha_s[2\alpha_l\Gamma+(1+\alpha_s^2)]r_s^{-1/2+\delta}\sin(1/2-\delta)\theta_s\},$$

$$\sigma_{xy} = \frac{K}{D\sqrt{2\pi}}\{2\alpha_l[2\alpha_s+\Gamma(1+\alpha_s^2)]r_l^{-1/2+\delta}\cos(1/2-\delta)\theta_l$$
$$-(1+\alpha_s^2)[2\alpha_l\Gamma+(1+\alpha_s^2)]r_s^{-1/2+\delta}\cos(1/2-\delta)\theta_s\}, \tag{35}$$

$$u_x = \frac{2K}{(1+2\delta)GD\sqrt{2\pi}}\{[2\alpha_s+\Gamma(1+\alpha_s^2)]r_l^{1/2+\delta}\sin(1/2+\delta)\theta_l$$
$$-\alpha_s[2\alpha_l\Gamma+(1+\alpha_s^2)]r_s^{1/2+\delta}\sin(1/2+\delta)\theta_s\},$$

$$u_y = \frac{2K}{(1+2\delta)GD\sqrt{2\pi}}\{\alpha_l[2\alpha_s+\Gamma(1+\alpha_s^2)]r_l^{1/2+\delta}\cos(1/2+\delta)\theta_l$$
$$-[2\alpha_l\Gamma+(1+\alpha_s^2)]r_s^{1/2+\delta}\cos(1/2+\delta)\theta_s\}, \tag{36}$$

where (r_l, θ_l) and (r_s, θ_s) are the distorted crack tip polar coordinates, defined by

$$z_l \equiv x+\mathrm{i}\alpha_l y \equiv r_l \mathrm{e}^{\mathrm{i}\theta_l}, \quad z_s \equiv x+\mathrm{i}\alpha_s y \equiv r_s \mathrm{e}^{\mathrm{i}\theta_s}. \tag{37}$$

Counterpart expressions for the lower half-plane, for which $-180^\circ \leqslant \theta_l, \theta_s \leqslant 0$, can be obtained from (35) and (36) by replacing Γ with $-\Gamma$.

We note that the only nonzero stress and displacement components along the bonded line are σ_{xy} and u_y, identical to the situation for the classic mode II problem. In fact, the angular stress and displacement variations described in (35) and (36) are only slightly different from those of the classic mode II problem, which can be seen from figures shown later. We also note that the normal stress σ_{yy} along the crack surfaces is found to equal $-\Gamma K r^{-1/2+\delta}\cos(\delta\pi)/(2\pi)^{1/2}$. Since σ_{yy} must be compressive there, as required by the contact condition, we must have $\Gamma K \geqslant 0$. When the crack faces are smooth, hence in frictionless

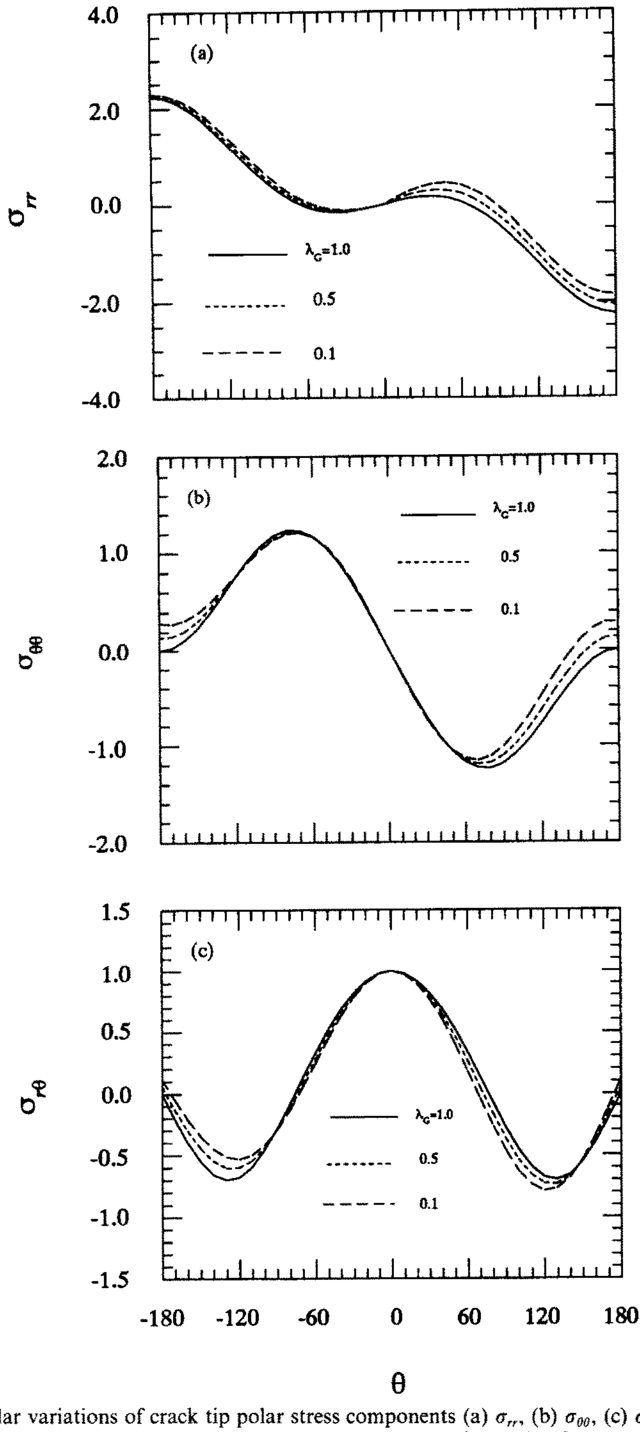

Fig. 2. Angular variations of crack tip polar stress components (a) σ_{rr}, (b) $\sigma_{\theta\theta}$, (c) $\sigma_{r\theta}$ for the case of $\nu_1 = \nu_2 = 0.3$, $\lambda_\rho = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$, with λ_G varying from 0.1 to 1.0.

contact, δ will be zero and the crack tip field solutions in (35) and (36) recover those presented in Deng (1993c), where an expression for the crack tip elastic energy release rate was also provided.

To illustrate how the singularity and angular distribution of the crack tip stress field vary with the coefficient of kinetic friction, crack speed and other key parameters of the bimaterial system, the angular variations of the crack tip polar stress components σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ for various cases are plotted in Figs 2–6, and the variations of the singularity index δ with respect to the system parameters under certain specified conditions are plotted in Fig. 7. In the figures, λ_G denotes the ratio of the shear modulus of material 1 to that of

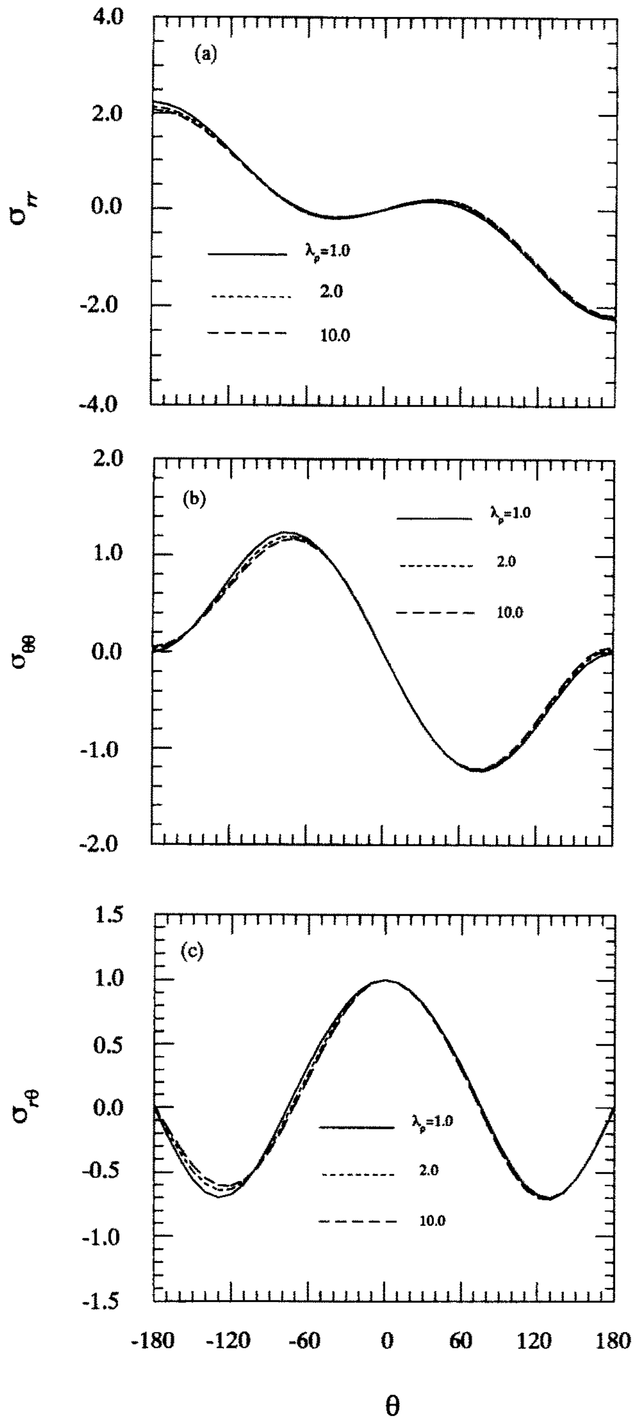

Fig. 3. Angular variations of crack tip polar stress components (a) σ_{rr}, (b) $\sigma_{\theta\theta}$, (c) $\sigma_{r\theta}$ for the case of $\nu_1 = \nu_2 = 0.3$, $\lambda_\mu = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$, with λ_ρ varying from 1.0 to 10.0.

material 2, λ_ρ the ratio of the mass density of material 1 to that of material 2, v/c_{s1} the ratio of crack growth speed to the shear wave speed of material 1, ν_1 the Poisson's ratio of material 1 (with that of material 2, ν_2, equal to 0.3), and μ_k the absolute value of the coefficient of kinetic friction between the contacting crack faces. The parameter values are so selected such that, in general, material 1 is "softer" (with smaller shear modulus) or "heavier" (with larger mass density) than material 2.

It is noted that the angle θ in Figs 2–6 is measured from the interface and is positive if counterclockwise, and all stress components are normalized by $Kr^{-1/2+\delta}/(2\pi)^{1/2}$, where r is

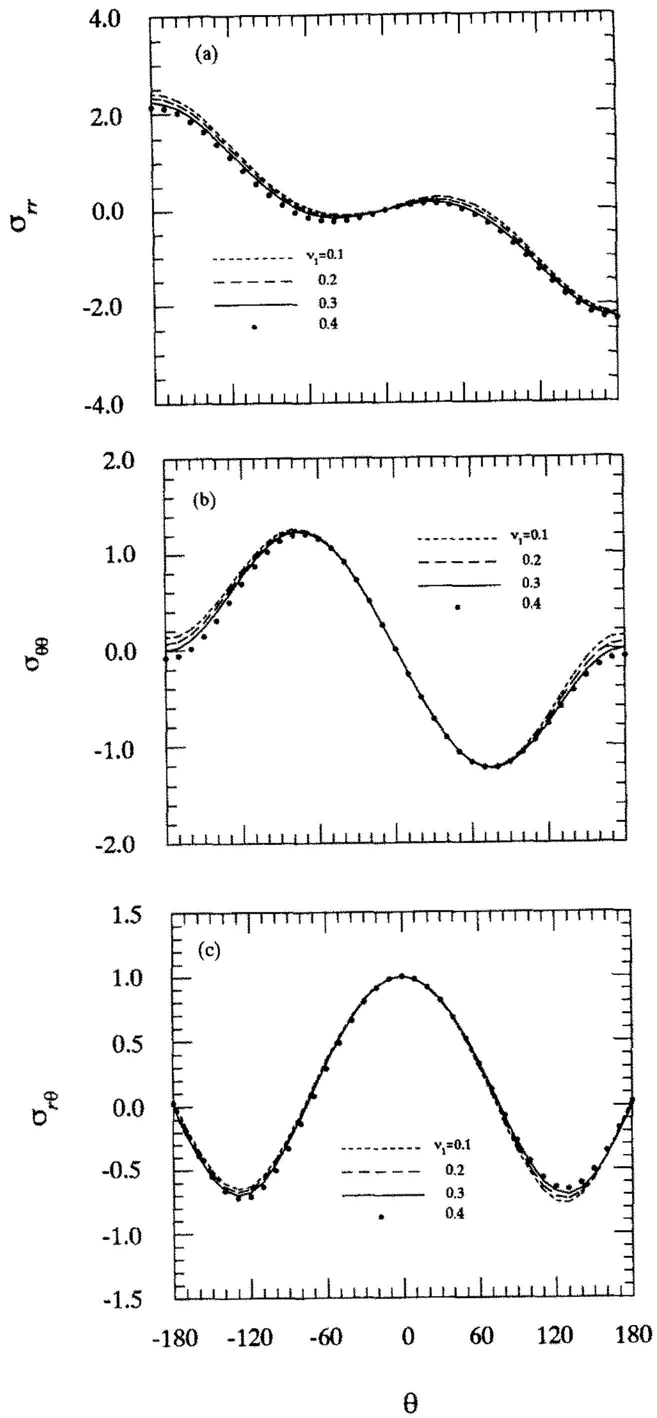

Fig. 4. Angular variations of crack tip polar stress components (a) σ_{rr}, (b) $\sigma_{\theta\theta}$, (c) $\sigma_{r\theta}$ for the case of $\nu_2 = 0.3$, $\lambda_G = 1.0$, $\lambda_\rho = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$, with ν_1 varying from 0.1 to 0.4.

the radial distance to the crack tip. Figure 2 shows the effect of λ_G on the stresses, Fig. 3 the effect of λ_ρ, Fig. 4 the effect of ν_1 with $\nu_2 = 0.3$, Fig. 5 the effect of v/c_{s1}, and Fig. 6 the effect of μ_k. It must be pointed out that the positive values observed in the figures for $\sigma_{\theta\theta}$ at $\theta = \pm 180$ imply that the values of the stress intensity factor K for the shown cases must be negative, hence the signs of the actual stress variations are just the opposite of those in the plots. For the parameter values examined, the angular variations of the stresses are found to change only slightly from one another. The largest changes occur when the shear moduli of the materials are very different (see Fig. 2) or when the crack speed is approaching

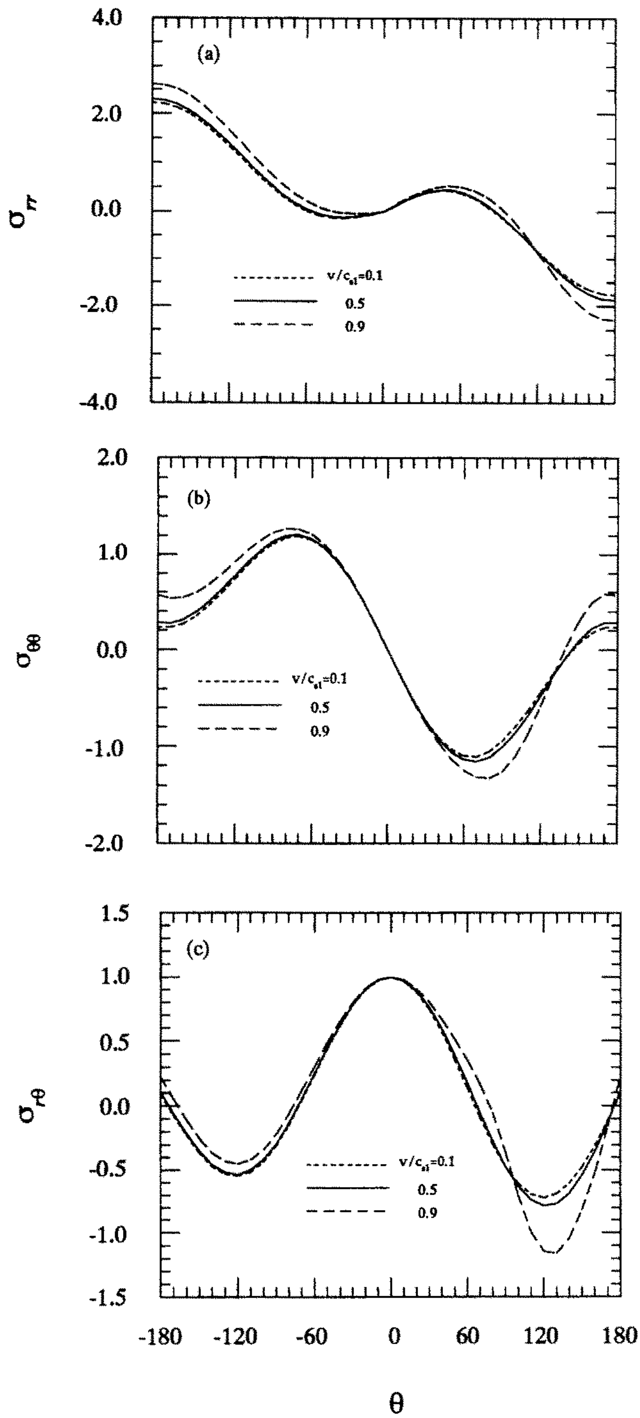

Fig. 5. Angular variations of crack tip polar stress components (a) σ_{rr}, (b) $\sigma_{\theta\theta}$, (c) $\sigma_{r\theta}$ for the case of $v_1 = v_2 = 0.3$, $\lambda_G = 0.1$, $\lambda_\rho = 1.0$ and $\mu_k = 0.4$, with v/c_{s1} varying from 0.1 to 0.9.

the Rayleigh wave speed of the "softer" material or material 1, which is about $0.927c_{s1}$ (see Fig. 5).

An important observation can be made from the above findings. It is clear from Figs 2–6 that, for a steadily growing interface crack in an isotropic bimaterial, the singular part of the circumferential stress $\sigma_{\theta\theta}$ at the crack tip is always positive in one solid and negative in the other (except near the crack flank), which implies that if the crack is to curve out of the interface, it is likely that it is going to branch into the solids with positive $\sigma_{\theta\theta}$. For example, for the cases shown in Fig. 2 with a softer material (for which $0° \leqslant \theta \leqslant 180°$) on

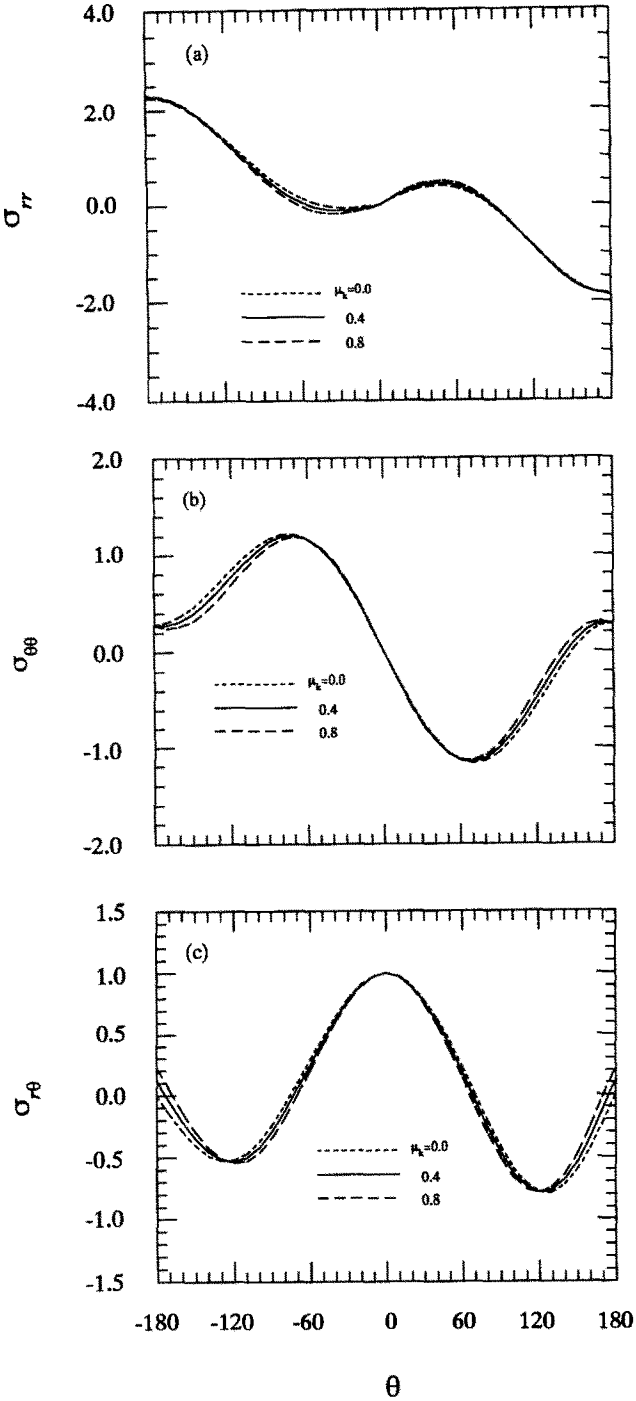

Fig. 6. Angular variations of crack tip polar stress components (a) σ_{rr}, (b) $\sigma_{\theta\theta}$, (c) $\sigma_{r\theta}$ for the case of $\nu_1 = \nu_2 = 0.3$, $\lambda_G = 0.1$, $\lambda_\rho = 1.0$ and $v/c_{s1} = 0.5$, with μ_k varying from 0.0 to 0.8.

top of a stiffer one (for which $-180^\circ \leqslant \theta \leqslant 0^\circ$), $\sigma_{\theta\theta}$ is found to be mostly positive within the softer material (note here that the stresses are normalized by a *negative* stress intensity factor K, along with other positive values). Hence, crack branching or kinking is more likely to occur towards the softer material, although whether it will actually take place will depend on the relative values of the fracture toughnesses of the interface and the component materials. Further, if the direction of crack branching is governed by the maximum of $\sigma_{\theta\theta}$, it will occur at an angle of about 70° from the interface, at which the shear stress $\sigma_{r\theta}$ is approximately zero, as indicated by the figures. For the cases studied, this branching angle

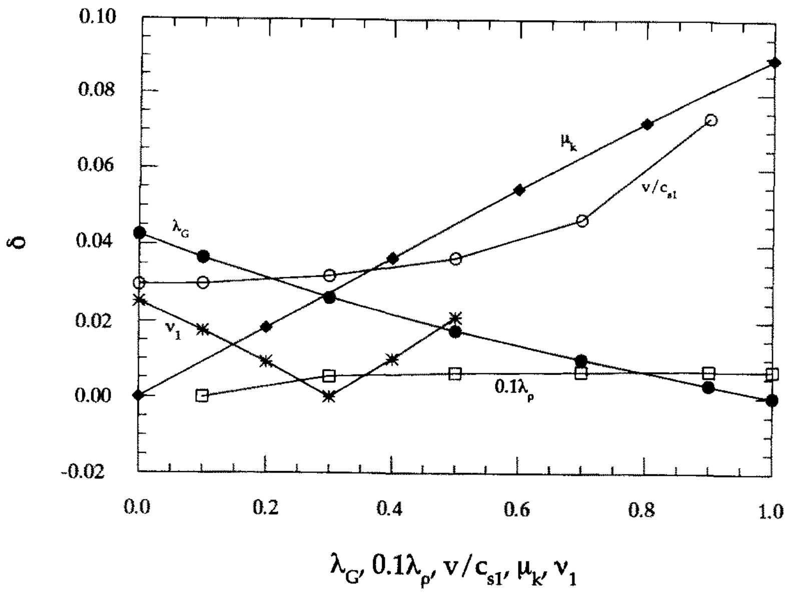

Fig. 7. Variations of the singularity index δ with respect to ν_1, λ_G, λ_p, v/c_{s1} and μ_k under certain specified conditions.

is found to vary only slightly when the system parameters change. The largest variation is observed when the crack propagation speed takes values from zero to $0.9c_{s1}$, which is near the Rayleigh wave speed or $0.927c_{s1}$ of the softer material and results in a crack branching angle of about 80 (see Fig. 5).

Before closing this section, we list here the specified conditions mentioned in Fig. 7 for the various curves. The δ vs λ_G curve is for the case $\nu_1 = \nu_2 = 0.3$, $\lambda_p = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$; the δ vs λ_p curve for the case $\nu_1 = \nu_2 = 0.3$, $\lambda_G = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$; the δ vs v/c_{s1} curve for the case $\nu_1 = \nu_2 = 0.3$, $\lambda_G = 0.1$, $\lambda_p = 1.0$ and $\mu_k = 0.4$; the δ vs μ_k curve for the case $\nu_1 = \nu_2 = 0.3$, $\lambda_G = 0.1$, $\lambda_p = 1.0$, and $v/c_{s1} = 0.5$; the δ vs ν_1 curve for the case $\nu_2 = 0.3$, $\lambda_G = 1.0$, $\lambda_p = 1.0$, $v/c_{s1} = 0.5$ and $\mu_k = 0.4$. It is clear from Fig. 7 that for the parameter values studied, the singularity index δ has values ranging from 0.0 to 0.1, which implies that the crack tip stress singularity for a growing interface crack with frictional crack surface contact will range from $r^{-0.5}$ to $r^{-0.4}$. The weakest singularities occur for cases with high crack propagation speed or large coefficient of kinetic friction, while the disparity between the mass densities of the materials is found to have a very limited effect on the crack tip singularity, making the singularity almost the same as $r^{-0.5}$.

5. FRICTIONAL CRACKS IN HOMOGENEOUS SOLIDS

An important aspect of the general solution of the interfacial crack tip fields given in the previous two sections is that it fully covers the case where the bimaterial is in fact homogeneous. To the author's best knowledge, there has been no published formal studies on the structure of the crack tip fields in a homogeneous anisotropic solids for a stationary or growing crack whose surfaces are in frictional contact, which, as we mentioned in the introductory section, is a situation that may arise in practice when a crack is in a nonuniform stress field where combined compression and shear loading around the crack may exist. As such, it is felt that a brief exposition here on the subject in question will be helpful.

When a bimaterial system becomes homogeneous, the hermitian matrix $\mathbf{H}$ associated with the system becomes real-valued. It is then easy to see from (18) that β, and hence δ, become zero, irrespective of the value of the kinetic friction coefficient μ_k. The general solution for $\mathbf{h}_1(z)$, given collectively in (15) and (19), now simplifies to

$$\mathbf{h}_1'(z) = \mathbf{h}_2'(z) = (2\pi z)^{-1/2}\xi(z)\mathbf{q}/2 + \xi^*(z)\mathbf{q}^* + \mathbf{g}(z)/2 \tag{38}$$

where $\mathbf{q}$ and $\mathbf{q}^*$ can be obtained from (17) with $\operatorname{Im} H_{12} = 0$. We remark that the analytic functions in (38) represent the complete crack tip fields for a crack with frictional or frictionless ($\mu_k = 0$) surface contact subjected to a combined compression and shear loading, and that these fields cannot be obtained directly from those of the classic mixed-mode problem, which can be seen from the comparison below for orthotropic materials.

To make further observations, we now focus our attention to homogeneous solids with at least orthotropic symmetries. In this case we note that $H_{12} = 0$ and the vectors $\mathbf{q}$ and $\mathbf{q}^*$ can now be written as

$$\mathbf{q} = (1,0), \quad \mathbf{q}^* = (-\mu_k, 1). \tag{39}$$

To compare the solution given by (38) and (39) with that for the classic mixed-mode problem, we list here a rearranged version of the mixed-mode crack tip field solution by Deng (1993a) for a general anisotropic material

$$\mathbf{h}_1'(z) = \mathbf{h}_2'(z) = (2\pi z)^{-1/2}\xi(z)(1,0)/2 + (2\pi z)^{-1/2}\xi^*(z)(0,1)/2 + \mathbf{g}(z)/2. \tag{40}$$

In the sense of a Williams-type series expansion, the first function in (40) represents mode II terms of order $r^{-1/2+n}$ ($n = 0, 1, 2, \ldots$), the second function mode I terms of order $r^{-1/2+n}$, and the third function mixed-mode terms of order r^n. It is clear that the second function in (38) differs from its counterpart in (40), hence (38) as a whole differs from (40), even when $\mu_k = 0$. Consequently, when compared with (40), a series expansion from (38) will have the same $r^{-1/2+n}$ mode II terms, but will lack those $r^{-1/2+n}$ mode I terms while gaining extra r^n terms. This stems from the fact that the classic mixed-mode problem assumes that the crack surfaces are not in contact, implying that the crack is subjected to combined tension and shear loading.

From the above comparisons, we can conclude that the crack tip stress and displacement fields for a crack with frictionless or frictional surface contact possess the same singularity as that for the classic mode II problem, with the same radial and angular variations and hence governed by the same mode II stress intensity factor or K_{II}. Since the singularity is of the order $r^{-1/2}$, the conventional contour integral for the crack tip strain energy release rate will be nonzero and in fact will be related to K_{II} in the usual way. When the crack surface contact is frictionless, we expect the contour integral to be path-independent. When the contact is frictional, the integral must be evaluated at the crack tip (more details in the next section).

6. ENERGY RELEASE RATE

The concept of energy release rate has played a central role in the development of fracture mechanics, and more recently, in the fracture mechanics of bimaterial interfaces. For elastic isotropic bimaterial systems, it has been observed that the critical value of the energy release rate for crack growth initiation, G_c, or interface toughness, is a monotonic function of the local mode mixity at the crack tip [for details see a review article by Hutchinson and Suo (1992)]. Specifically, the interface toughness G_c is minimum in a tensile state and increases continuously as shear stress increases, and climbs to extreme values when the stress state is mostly shear ahead of the crack tip. It is noted that the above observations are made from interpretations of various experimental data based on the singular crack tip fields for interface cracks without crack surface contact, namely the usual inverse-square-root singularity with or without oscillation. An important aspect of this type of singularity is that the energy release rate will have a finite value related to the stress intensity factors of the singular crack tip fields, and can be evaluated numerically for complex problems using a path-independent contour integral, namely the well-known J-integral (Rice, 1968).

The strong dependence of interface toughness on mode mixity has been attributed to several factors, including asperity contact of the crack faces at a certain distance behind the crack tip and plastic deformation around the crack tip (Evans *et al.*, 1990). More recently, a study by Stringfellow and Freund (1993) reveals that another key mechanism

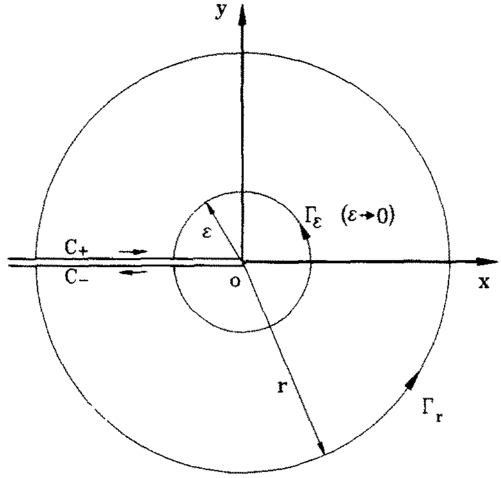

Fig. 8. Crack tip contours used for the calculation of the energy release rate.

for increased interface toughness for shear mode crack growth initiation is the sliding friction between the crack faces, which may develop in a continuous contact zone immediately behind the crack tip when the crack tip is in a local shear state. It is found that the existence of sliding friction between the contacting crack faces reduces substantially the value of the stress intensity factor for a crack in a homogeneous solid and causes a significant drop in the energy release rate evaluated at the tip of an interface crack. Again, it is noted, that the preceding study presumes the existence of an inverse-square-root singularity at the crack tip.

The purpose of this section is to apply the findings of the asymptotic analyses of previous sections to give an account of the observed reduction in the energy release rate for the cases studied by Stringfellow and Freund (1993). In particular, the effect of a weaker singularity at the crack tip on the calculation of the energy release rate will be noted. When the energy release rate is calculated from a J-integral based contour integral, explicit expressions for the path dependence of this integral near the crack tip will be derived.

We consider a crack which is growing steadily (quasi-statically or dynamically), or a crack which is stationary under quasi-static loading. As shown in Fig. 8, there is a region around the crack tip that is enclosed by line segments C_+ and C_- along the crack flank and circular contours Γ_ε and Γ_r, both centered at the crack tip and with radii r and $\varepsilon \to 0$, respectively. With the crack tip excluded from this region, as implied by the limit that the radius of Γ_ε is approaching zero rather than equals zero, the stress and displacement fields and their spatial derivatives will be smooth functions within this region, hence enabling the use of the divergence theorem. Under the above settings, it can be shown that the following contour integral along the complete path $\Gamma_r + (-\Gamma_\varepsilon) + C_+ + C_-$ is zero:

$$\oint_{\Gamma_r - \Gamma_\varepsilon + C_+ + C_-} [(W+L)n_1 - \sigma_{ij} n_j \partial u_i / \partial x] \, \mathrm{d}S = 0 \tag{41}$$

where the indicial summation convention is used, with Latin indices taking values 1 or x and 2 or y; n_i is the outward unit normal vector with the region on the left of the path direction considered the interior; and $W = \sigma_{ij}\varepsilon_{ij}/2$ and $L = \rho(\mathrm{d}u_i/\mathrm{d}t)(\mathrm{d}u_i/\mathrm{d}t)/2$ are, respectively, the strain energy density and kinetic energy density. It is important to note that although the conclusion shown in (41) is expressed in terms of circular contours, its validity also holds for contours of any other shape. An immediate implication of eqn (41) can be written as

$$G_r = G_\varepsilon + \Delta G_r,$$

$$G_r = \int_{\Gamma_r} [(W+L)n_1 - \sigma_{ij} n_j \partial u_i / \partial x] \, \mathrm{d}S,$$

$$G_\varepsilon = \int_{\Gamma_\varepsilon} [(W+L)n_1 - \sigma_{ij} n_j \partial u_i/\partial x]\,\mathrm{d}S,$$

$$\Delta G_r = -\int_{C_+ + C_-} [(W+L)n_1 - \sigma_{ij} n_j \partial u_i/\partial x]\,\mathrm{d}S, \tag{42}$$

where G_ε is the commonly known energy release rate (at the crack tip), and G_r is the rate of energy being released into the region surrounded by Γ_r, which differs from G_ε by ΔG_r. Noting that $n_1 = 0$ along the crack flank and that traction $(\sigma_{xy}, \sigma_{yy})$ and normal displacement component u_y are continuous across the crack surfaces, one can simplify the expression for ΔG_r into

$$\Delta G_r = -\int_0^r \sigma_{xy}(r, \pm\pi) \frac{\partial}{\partial x}[u_x(r,\pi) - u_x(r, -\pi)]\,\mathrm{d}r. \tag{43}$$

To arrive at explicit expressions for the path dependence of the contour integral G_r near the crack tip for various cases, the integrals G_ε and ΔG_r will be evaluated using only the leading nonzero terms of the stress and displacement field expansions at the crack tip.

Interface cracks

For interface cracks with frictional contact, the crack tip singularity will be of the form $r^{-1/2+\delta}$. The discussion here will be limited to cases where δ is positive, which is true for crack growth with crack tip field in a local steady state, and for crack growth initiation under continuous loading conditions. It is easy to see that this type of singularity will lead to an integrand for G_ε that is as singular as $r^{-1+2\delta}$, which, when combined with the relation $\mathrm{d}S = r\,\mathrm{d}\theta$ along a circular path, will result in a zero value for G_ε since ε is approaching zero. To evaluate ΔG_r, we note from the general solution in Section 3 that the leading nonzero term is the singular term, which generates the following stress and displacement components along the crack flank:

$$\sigma_{xy}(r, \pm\pi) = r^{-1/2+\delta}(\mu_k \beta K)\cos(\pi\delta)/\sqrt{2\pi},$$

$$\frac{\partial}{\partial x}[u_x(r,\pi) - u_x(r,-\pi)] = -r^{-1/2+\delta} K \cos(\pi\delta)|\mathbf{H}|/H_{22}\sqrt{2\pi}. \tag{44}$$

Substitution of (44) into (43) and (42) then yields the following expressions for ΔG_r and G_r in a region near the crack tip:

$$G_r = \Delta G_r = \left[\frac{r^{2\delta}(\mu_k\beta)\cos^2(\pi\delta)}{\pi\delta}\right]\left[\frac{K^2|\mathbf{H}|}{4H_{22}}\right]. \tag{45}$$

Since $\mu_k \beta > 0$, the above equation always gives a positive value for G_r at $r > 0$.

For interface cracks with frictionless contact, δ is zero and G_ε is finite. Since there is no friction, hence no shear stress along the crack flank, the contour integral G_r will be path-independent, as indicated by a zero ΔG_r value. Expressions for G_r in terms of the stress intensity factor K can be found in Deng (1993b,c) for various cases. It is noted that the final expressions for G_r for these cases can be derived from that in (45) by noting the relation in (18) and by taking the limit $\mu_k \to 0$, hence the limit $\delta \to 0$.

Cracks in homogeneous solids

For cracks with frictionless or frictional surface contact in homogeneous solids, the singularity is of the inverse-square-root type, which will give rise to a nonzero value for G_ε. Since the leading and singular term of the crack tip fields do not depend on the amount of friction, expressions derived in Deng (1993b,c) for interface cracks with frictionless contact can be applied here for G_r, with the upper and lower half-planes sharing the same elastic constants. Along the crack flank, the leading nonzero term for u_x is governed by K, which gives

$$\frac{\partial}{\partial x}[u_x(r,\pi)-u_x(r,-\pi)] = -r^{-1/2}K|\mathbf{H}|/H_{22}\sqrt{2\pi}. \tag{46}$$

However, the K-controlled singular crack tip fields produce zero tractions along the crack flank, irrespective of the amount of friction between the crack faces. This can be argued from the expression for σ_{yy} in (22) for a general interface crack. When a bimaterial becomes homogeneous, the mismatch parameter β becomes zero, which leads to zero σ_{yy}, and hence zero σ_{xy} values along the crack flank. As such, the leading nonzero term for the tractions must come from the constant terms of the entire function $\xi^*(z)$ and those in $\mathbf{g}(z)$, but the latter happen to produce zero tractions along the entire x-axis, as discussed in Section 3. Therefore, the leading nonzero terms for the tractions along the crack flank will only come from the constant term of the entire function $\xi^*(z)$, say $\xi^*(z) = C$, which results in

$$\begin{aligned}\sigma_{xy} &= -2C\mu_k|\mathbf{H}|/H_{11}H_{22},\\ \sigma_{yy} &= 2C|\mathbf{H}|/H_{11}H_{22}.\end{aligned} \tag{47}$$

Since the crack surfaces must be in contact, hence σ_{yy} be nonpositive, C must be negative for nontrivial cases. Substitution of (46) and (47) into (43) and (42) yields

$$\begin{aligned}G_r &= G_\varepsilon + \Delta G_r,\\ \Delta G_r &= 4(-C)(\mu_k K)|\mathbf{H}|^2\sqrt{r/2\pi}/(H_{11}H_{22}^2).\end{aligned} \tag{48}$$

As argued in Section 3 for the more general case of an interface crack, the sign of μ_k will be the same as that of K, hence ΔG_r is always positive for $\mu_k \neq 0$ at $r > 0$, which means that the contour integral G_r has the energy release rate G_ε as its local minimum at the crack tip. When friction is not present, G_r will be path-independent, which is apparent from (48) when $\mu_k = 0$.

Discussion

Results obtained above will now be used to discuss recent findings by Stringfellow and Freund (1993) regarding residual-stress induced delamination of thin films from their substrates. The steady-state delamination process modelled in their study can be viewed here as a limiting case of our analysis, namely the steady-state, quasi-static growth of a crack along the interface of an elastic isotropic bimaterial. Results for this case can be obtained from those for a propagating crack by letting the crack speed approach zero in pertinent end expressions.

The first type of problem studied by Stringfellow and Freund (1993) deals with cracks with frictional contact along interfaces of two identical solids. As shown by the asymptotic analyses in previous sections, the singular crack tip fields are the same as those for a classic mode II problem and hence governed by the mode II stress intensity factor K_{II}, regardless of the amount of friction between the contacting crack faces, which implies that the energy release rate G_ε will be related to K_{II} in the same way, namely $G_\varepsilon = K^2/E^*$, where $E^* = E$, the Young's modulus, in the plane stress, and $E^* = E/(1-\nu^2)$ in plane strain. In light of this, if the energy release rate G_ε is evaluated with the J-integral along a contour very close to the crack tip, as is done in the study in question, the difference between G_ε for a crack without friction and that for a crack with friction will equal the difference between squares of the corresponding values of K_{II}. According to eqn (48) however, if the contour is not that close to the crack tip, the calculated value for G_ε from a contour integral for a crack with friction is in fact the value of G_r, increased by an amount of approximately ΔG_r. In the study referred to here, the drop in K_{II} from a crack without friction to one with friction

Table 1

E_f/E_s	0.25	0.25	0.5	0.5
μ_k	0.5	1.0	0.5	1.0
δ	0.02721	0.05404	0.01515	0.03023

correlates well with the corresponding drop in the energy release rate [see Figs 7a and 11 in Stringfellow and Freund (1993)], implying that ΔG_r is quite small. One reason for a small ΔG_r contribution is that the innermost contour (small r) was used for calculating the J-integral and that ΔG_r decays fairly rapidly as r decreases. It is plausible that another reason is that the constant C in (48), which for the present case equals half of the compressive normal stress value between the crack faces, is small.

The second type of problem studied by Stringfellow and Freund (1993) is concerned with the extension of the above analysis to the case of cracks along the interfaces of dissimilar materials. Theoretically, the crack tip singularity is weaker, although slightly, than that of the classic mode II problem and the energy release rate is exactly zero at the crack tip. However, numerically the crack tip cannot be actually reached, and an integration contour somewhat away from the crack tip is usually used. While this contour may be able to give a converged value for a theoretically nonzero energy release rate, say, for the case of a crack in a homogeneous solid, it may not be able to give a converged zero value for the present case, as in the referred study here. A reason for this disparity in performance is that contributions to the integral from secondary factors decay at different rates for different cases. Accordingly, if the energy release rate is evaluated numerically using the J-integral, a nonzero value will usually be produced. In general, the computed value of this integral will be somewhat random, depending on how far the contour is from the crack tip. When the contour is circular and is sufficiently close to the crack tip, the dependence of the integral will be given by the expression for G_r, as shown in (45). To this end, it is important to note that G_r has a weak dependence on r, meaning that ΔG_r, the contribution to G_r from friction along the crack flank, a secondary factor, decays slowly as r decreases. This is in strong contrast with problems discussed a little earlier for cracks in homogeneous solids, where ΔG_r varies with $r^{0.5}$. In comparison, ΔG_r varies with $r^{2\delta}$ here, with δ of the order of 0.05. For example, for the case of a bimaterial system with homogeneous Poisson's ratio of 0.3, the variation of δ with μ_k and E_f/E_s (where E_f for the film and E_s for the substrate are the Young's moduli) is as in Table 1. On the other hand, although the calculated G_r value is nonzero, it is expected that when a contour quite close to the crack tip is used, the G_r value will still be significantly below its level for the case of no friction between the crack faces.

In conclusion, for interface cracks with frictional crack surface contact, the energy release rate is not a well defined fracture parameter, because theoretically it is zero and its value from J-integral calculation is path-dependent. In this case, the stress intensity factor K appears to be a better measure of crack tip fracture driving force for crack growth and its initiation. Nonetheless, as a qualitative indicator, the energy release rate calculated along a finite contour near the crack tip is still a useful tool for illustrating the effect of friction on interface toughness, as demonstrated by Stringfellow and Freund (1993) in their newly published report.

7. SUMMARY

An asymptotic analysis of a stationary or steadily growing interface crack with frictional contact in an elastic anisotropic bimaterial has been presented, which requires that the solid is so oriented that the plane perpendicular to the crack surface is a material symmetry plane. A full representation of plane strain and plane stress crack tip fields is provided in terms of several arbitrary entire functions, which can be used to generate complete Williams-type series expansions for the crack tip fields. As special cases, the above general solutions also apply to problems with cracks along the interfaces of isotropic bimaterials or cracks in homogeneous solids. The main points of this research can be summarized as follows.

First, the singular part of the frictional interface crack tip fields is governed by only one stress intensity factor and is of the type $r^{-1/2+\delta}$, where the singularity index δ can in general be positive or negative depending on deformation history of the crack flank near the crack tip. However, when the loading is increased monotonically from zero or when the crack is growing steadily near the crack tip, δ must always be positive, implying that the crack tip singularity is weaker than the conventional inverse-square-root singularity. When crack contact is frictionless or when the bimaterial becomes homogeneous, δ will be zero.

Second, for solids with at least orthotropic symmetries, the singular crack tip fields are similar to those of the classic mode II problem. When these types of anisotropic bimaterials become homogeneous, the singular crack tip fields will be identical to those of the classic mode II problem, regardless of the amount of sliding friction between the contacting crack faces. Further, the complete crack tip fields represented by the arbitrary entire functions now fully describe the near-tip behaviour of the homogeneous materials under combined compression and shear loading conditions. These complete crack tip fields cannot be obtained directly from those for the classic mixed-mode problem.

Third, if crack branching or kinking away from the interface is to occur in a bimaterial with homogeneous Poisson's ratio, it is more likely that the crack will go into the material with lower shear modulus than into the one with higher shear modulus. From the angular location of the maximum circumferential stress $\sigma_{\theta\theta}$, the angle at which the crack breaks away from the interface is estimated to be near 70°, where the shear stress $\sigma_{r\theta}$ is approximately zero.

Finally, for cases where the singularity of the crack tip fields is weaker than $r^{-1/2}$, such as during steady-state interfacial crack growth with sliding crack surface friction, the stress intensity factor is a better choice as a measure of the crack tip fracture driving force than the energy release rate. This is because theoretically the energy release rate is zero at the crack tip and estimates of the energy release rate computed from finite element data using contour integrals, such as the J-integral, are arbitrary and not reliable quantitatively, in the sense that the estimated values will depend on the location of the particular integration contour used. However, it is felt that a fracture criterion based on the energy release rate has one strong advantage over a stress intensity factor based one, namely that a properly posed energy criterion can provide a unified dimensional measure for the crack tip state and for the fracture toughness of the material, as opposed to the K based criterion that changes its definition and dimension from one class of problems to the other, depending on the singularity of the crack tip fields. As such, it is hoped that the findings of this study can act as a basis for a modified, energy release rate based fracture criterion for interface cracks.

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